THE 4^{TH} MOMENT OF THE VELOCITY DISTRIBUTION FOR AN ANISOTROPIC SPHERICALLY SYMMETRIC DISTRIBUTION.

FOR A NON-ROTATING SPHERICALLY SYMMETRIC SELF-GRAVITATING SYSTEM, THE VLASOV EQUATION CAN BE WRITTEN

$$\frac{Df}{dt} = v_r \frac{\partial f}{\partial r} + \left(\frac{v_\theta^2 + v_\phi^2}{r} - \frac{d\Phi}{dr}\right)\frac{\partial f}{\partial v_r} + \frac{1}{r}\left(v_\phi^2 \cot\theta - v_r v_\theta\right)\frac{\partial f}{\partial v_\theta} - \frac{1}{r}\left(v_\phi v_r + v_\phi v_\theta \cot\theta\right)\frac{\partial f}{\partial v_\phi} = 0 \qquad (1)$$

WE'RE INTERESTED IN THE FOURTH MOMENT OF THE VELOCITY, AND SO MULTIPLY (1) BY $(v_{r,\theta,\phi}\, v_{r,\theta,\phi}\, v_{r,\theta,\phi})$ AND INTEGRATE OVER VELOCITY SPACE. OF THE 10 POSSIBLE MOMENTS GENERATED IN THIS WAY, THE INTERESTING ONES ARE:

$$\int (1)\, v_r^3 d^3v \Rightarrow \frac{d}{dr}\left(\rho \overline{v_r^4}\right) + 3\rho \overline{v_r^2}\frac{d\Phi}{dr} - \frac{3\rho}{r}\overline{v_r^2(v_\theta^2+v_\phi^2)} + \frac{2\rho}{r}\overline{v_r^4} = 0 \qquad (2)$$

$$\int (1)\, v_r v_\theta^2 d^3v \Rightarrow \frac{d}{dr}\left(\rho \overline{v_r^2 v_\theta^2}\right) + \rho \overline{v_\theta^2}\frac{d\Phi}{dr} - \frac{\rho}{r}\left(\overline{v_\theta^4} + \overline{v_\theta^2 v_\phi^2}\right) + \frac{4\rho}{r}\overline{v_\theta^2 v_r^2} = 0 \qquad (3)$$

FOR A SPHERICALLY SYMMETRIC SYSTEM, THERE ARE ONLY 2 INDEPENDENT COMPONENTS OF VELOCITY, v_r AND $v_t = (v_\theta^2 + v_\phi^2)^{1/2}$. TO CONVERT THE TERMS WITH v_θ AND v_ϕ IN TO v_t, WE WRITE THE VELOCITY PHASE SPACE AT ANY POINT IN SPHERICAL POLAR COORDINATES:

$$\left.\begin{array}{l} \left.\begin{array}{l} v_\theta = v \sin\xi \sin\eta \\ v_\phi = v\cos\xi \sin\eta \end{array}\right\} \Rightarrow v_t = v\sin\eta \\ v_r = v\cos\eta \end{array}\right\} \quad d^3v = v^2 \sin\eta \, d\eta \, d\xi \, dv$$

$$\Rightarrow \rho\overline{v_\theta^4} = \iiint v^4 \sin^4\xi \sin^4\eta \, f \cdot v^2 \sin\eta \, d\eta\, d\xi\, dv$$

$$= \int_{\xi=0}^{2\pi} \frac{1}{2\pi}\sin^4\xi \, d\xi \cdot 2\pi \iint v^4 \sin^4\eta \, f \, v^2 \sin\eta \, d\eta \, dv$$

$$= \frac{3}{8} \cdot \rho\overline{v_t^4}$$

AND SIMILARLY, $\rho\overline{v_\phi^4} = \frac{3}{8}\rho\overline{v_t^4}$; $\rho\overline{v_\theta^2 v_\phi^2} = \frac{1}{8}\rho\overline{v_t^4}$

$$\overline{v_\theta^2} = \overline{v_\phi^2} = \frac{1}{2}\overline{v_t^2} \; ; \; \overline{v_r^2 v_\theta^2} = \overline{v_r^2 v_\phi^2} = \frac{1}{2}\overline{v_r^2 v_t^2}$$

SUBSTITUTING THIS INFORMATION INTO EQUATIONS (2) + (3)

$$\Rightarrow \quad \frac{d}{dr}\left(\rho\overline{v_r^4}\right) + 3\rho\overline{v_r^2}\frac{d\Phi}{dr} - \frac{3\rho}{r}\overline{v_r^2 v_t^2} + \frac{2\rho}{r}\overline{v_r^4} = 0 \qquad (4)$$

$$\frac{d}{dr}\left(\rho\overline{v_r^2 v_t^2}\right) + \rho\overline{v_t^2}\frac{d\Phi}{dr} - \frac{\rho}{r}\overline{v_t^4} + \frac{4\rho}{r}\overline{v_r^2 v_t^2} = 0 \qquad (5)$$

[CHECK: FOR AN ISOTROPIC SYSTEM, IT TURNS OUT THAT $\overline{v_r^2 v_t^2} = \frac{2}{3}\overline{v_r^4}$ AND $\overline{v_t^4} = \frac{8}{3}\overline{v_r^4}$

SO (4) $\Rightarrow$ $\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + 3\rho\overline{v_r^2}\frac{d\Phi}{dr} = 0$

AND (5) $\Rightarrow$ $\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + \frac{3}{2}\rho\overline{v_t^2}\frac{d\Phi}{dr} = 0$

I.E. AS $\overline{v_t^2} = \overline{(v_\theta^2 + v_\phi^2)} = 2\overline{v_r^2}$ FOR AN ISOTROPIC SYSTEM, THESE 2 EQUATIONS ARE DEGENERATE (AND RIGHT I THINK – SEE RESEARCH EXAM)]

TO ELIMINATE $\overline{v_r^2 v_t^2}$ BETWEEN (4) AND (5),

$$(4) \Rightarrow \rho\overline{v_r^2 v_t^2} = \frac{r}{3}\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + \rho\overline{v_r^2}\, r\frac{d\Phi}{dr} + \frac{2}{3}\rho\overline{v_r^4}$$

SUBSTITUTING INTO (5)

$$\Rightarrow \frac{d}{dr}\left[\frac{r}{3}\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + \rho\overline{v_r^2}\, r\frac{d\Phi}{dr} + \frac{2}{3}\rho\overline{v_r^4}\right] + \rho\overline{v_t^2}\frac{d\Phi}{dr} - \frac{\rho}{r}\overline{v_t^4} + \frac{4}{3}\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + 4\rho\overline{v_r^2}\frac{d\Phi}{dr} + \frac{8}{3}\frac{1}{r}\rho\overline{v_r^4} = 0$$

$$\Rightarrow \rho\overline{v_t^4} = \frac{r^2}{3}\frac{d^2}{dr^2}\left(\rho\overline{v_r^4}\right) + \frac{7}{3}r\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + \frac{8}{3}\rho\overline{v_r^4}$$

$$+ r^2\frac{d\Phi}{dr}\frac{d}{dr}\left(\rho\overline{v_r^2}\right) + \left(5\rho\overline{v_r^2} + \rho\overline{v_t^2}\right) r\frac{d\Phi}{dr} + \rho\overline{v_r^2}\, r^2\frac{d^2\Phi}{dr^2}$$

BUT THE SECOND MOMENT EQUATION (HYDROSTATIC EQUILIBRIUM) YIELDS

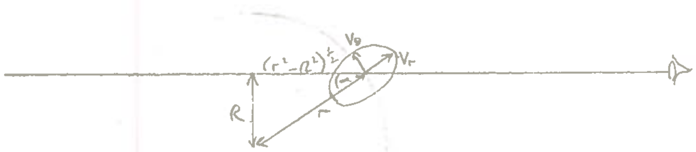
$$\frac{d}{dr}\left(\rho\overline{v_r^2}\right) + \frac{\rho}{r}\left[2\overline{v_r^2} - \overline{v_t^2}\right] + \rho\frac{d\Phi}{dr} = 0$$

$$\Rightarrow \rho\overline{v_t^4} = \frac{r^2}{3}\frac{d^2}{dr^2}\left(\rho\overline{v_r^4}\right) + \frac{7}{3}r\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + \frac{8}{3}\rho\overline{v_r^4}$$

$$+ \left\{\left(3\rho\overline{v_r^2} + 2\rho\overline{v_t^2}\right) r\frac{d\Phi}{dr} - \rho r^2\left(\frac{d\Phi}{dr}\right)^2 + \rho\overline{v_r^2}\, r^2\frac{d^2\Phi}{dr^2}\right\} \qquad (6)$$

THE QUANTITY WHICH CAN, IN PRINCIPLE, BE OBSERVED IN THIS SYSTEM IS THE PROJECTED LINE-OF-SIGHT 4TH VELOCITY MOMENT.

ONCE AGAIN FOLLOWING BINNEY AND TREMAINE (PP 207-8) THE PROJECTED 4TH MOMENT IS GIVEN BY



$$\mu(R)\overline{v_{LOS}^4} = 2\int_R^\infty \rho(r)\,\overline{\left(v_r\cos\alpha - v_\theta\sin\alpha\right)^4}\,\frac{r\,dr}{(r^2-R^2)^{\frac{1}{2}}}$$

$$= 2\int_R^\infty \rho(r)\left[\overline{v_r^4}\cos^4\alpha + 6\overline{v_r^2 v_\theta^2}\cos^2\alpha\sin^2\alpha + \overline{v_\theta^4}\sin^4\alpha\right] \times \frac{r\,dr}{(r^2-R^2)^{\frac{1}{2}}}$$

BUT $\cos^2\alpha = \frac{r^2-R^2}{r^2}$; $\sin^2\alpha = \frac{R^2}{r^2}$ AND $\overline{v_\theta^4} = \frac{3}{8}\overline{v_t^4}$; $\overline{v_r^2 v_\theta^2} = \frac{1}{2}\overline{v_r^2 v_t^2}$

$$\Rightarrow \mu(R)\overline{v_{LOS}^4} = 2\int_R^\infty \left[\frac{(r^2-R^2)^{3/2}}{r^3}\rho\overline{v_r^4} + 3\frac{R^2}{r^3}(r^2-R^2)^{\frac{1}{2}}\rho\overline{v_r^2 v_t^2} + \frac{3}{8}\frac{R^4}{r^3}(r^2-R^2)^{-\frac{1}{2}}\rho\overline{v_t^4}\right] dr$$

SUBSTITUTING FOR $\rho\overline{v_r^2 v_t^2}$ FROM (4) AND $\rho\overline{v_t^4}$ FROM (6)

$$\Rightarrow \mu(R)\overline{v_{LOS}^4} = 2\int_R^\infty \left\{\frac{(r^2-R^2)^{3/2}}{r^3}\rho\overline{v_r^4} + \frac{R^2}{r^2}(r^2-R^2)^{\frac{1}{2}}\left[\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + 3\rho\overline{v_r^2}\frac{d\Phi}{dr} + \frac{2\rho}{r}\overline{v_r^4}\right]\right.$$

$$+ \frac{3}{8}\frac{R^4}{r^3}(r^2-R^2)^{-\frac{1}{2}}\left[\frac{1}{3}r^2\frac{d^2}{dr^2}\left(\rho\overline{v_r^4}\right) + \frac{7}{3}r\frac{d}{dr}\left(\rho\overline{v_r^4}\right) + \frac{8}{3}\rho\overline{v_r^4}\right.$$

$$\left.\left. + \left(3\rho\overline{v_r^2} + 2\rho\overline{v_t^2}\right)r\frac{d\Phi}{dr} - \rho r^2\left(\frac{d\Phi}{dr}\right)^2 + \rho\overline{v_r^2}\, r^2\frac{d^2\Phi}{dr^2}\right]\right\} dr \qquad (7$$

RE-ARRANGING SO ALL THE "KNOWN QUANTITIES" ARE ON THE LHS

$$\Rightarrow \mu(R)\,\overline{v^4_{LOS}} - 2\int_R^\infty \left(\frac{3R^2}{r^2}(r^2-R^2)^{\frac{1}{2}}\rho\overline{v_r^2}\frac{d\Phi}{dr} + \frac{3}{8}\frac{R^4}{r^3}(r^2-R^2)^{-\frac{1}{2}}\left[(3\rho\overline{v_r^2}+2\rho\overline{v_t^2})\,r\frac{d\Phi}{dr} - \rho r^2\left(\frac{d\Phi}{dr}\right)^2 + \rho\overline{v_r^2}\,r^2\frac{d^2\Phi}{dr^2}\right]\right) dr$$

$$= 2\int_R^\infty \left(\frac{1}{8}\frac{R^4}{r}(r^2-R^2)^{-\frac{1}{2}}\frac{d^2}{dr^2}(\rho\overline{v_r^4}) + \frac{R^2}{r^2}(r^2-R^2)^{-\frac{1}{2}}\left(r^2-\frac{1}{8}R^2\right)\frac{d}{dr}(\rho\overline{v_r^4}) + r(r^2-R^2)^{-\frac{1}{2}}\rho\overline{v_r^4} \right) dr \qquad (8)$$

SUBSTITUTING $x = r^2$, $X = R^2$ $\Rightarrow$ $dr = \frac{1}{2}\frac{dx}{x^{\frac{1}{2}}}$; $\frac{d}{dr} = 2x^{\frac{1}{2}}\frac{d}{dx}$; $\frac{d^2}{dr^2} = 4x\frac{d^2}{dx^2} + 2\frac{d}{dx}$

AND WRITING LHS = $H(X)$ A MEASURABLE QUANTITY

$$\Rightarrow H(X) = \int_X^\infty (x-X)^{-\frac{1}{2}}\left[\frac{1}{2}X^2\frac{d^2\rho\overline{v_r^4}}{dx^2} + 2X\frac{d\rho\overline{v_r^4}}{dx} + \rho\overline{v_r^4}\right] dx$$

INTEGRATING THE LAST TERM BY PARTS

IF $\rho\overline{v_r^4} \to 0$ FASTER THAN $x^{-\frac{1}{2}}$ AS $x \to 0$ (TO BE CHECKED A POSTERIORI)

$$= \int_X^\infty (x-X)^{-\frac{1}{2}}\left[\frac{1}{2}X^2\frac{d^2\rho\overline{v_r^4}}{dx^2} + 2X\frac{d}{dx}\rho\overline{v_r^4}\right] dx + \left[\cancelto{0}{\rho\overline{v_r^4}\cdot 2(x-X)^{\frac{1}{2}}} - \int 2(x-X)^{\frac{1}{2}}\cdot\frac{d}{dx}\rho\overline{v_r^4}\,dx\right]_X^\infty$$

$$= \int_X^\infty (x-X)^{-\frac{1}{2}}\left[\frac{1}{2}X^2\frac{d^2}{dx^2}\rho\overline{v_r^4} + 2(2X-x)\frac{d}{dx}\rho\overline{v_r^4}\right] dx$$

WRITING $y = \frac{d}{dx}(\rho\overline{v_r^4})$

$$= \int_X^\infty (x-X)^{-\frac{1}{2}}\left[\frac{1}{2}X^2\frac{dy}{dx} + 2(2X-x)y\right] dx$$

BY PARTS AGAIN

IF $y \to 0$ FASTER THAN $x^{-3/2}$ AS $x \to \infty$ (I.E. SAME CRITERION AS ABOVE)

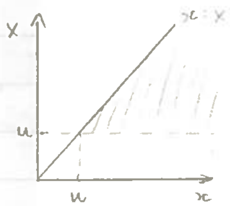
$$= \int_X^\infty (x-X)^{-\frac{1}{2}}\frac{1}{2}X^2\frac{dy}{dx}\,dx + 2\left[\cancelto{0}{y\cdot\frac{2}{3}(-x+4X)}(x-X)^{\frac{1}{2}} - \int\frac{2}{3}(-x+4X)(x-X)^{\frac{1}{2}}\frac{dy}{dx}\,dx\right]_X^\infty$$

$$= \int_X^\infty (x-X)^{-\frac{1}{2}}\left(\frac{4}{3}x^2 - \frac{20}{3}Xx + \frac{35}{6}X^2\right)\frac{dy}{dx}\,dx$$

NOW TO ABEL INVERT THIS:

MULTIPLYING BY $(X-u)^{-1/2}$ AND INTEGRATING $\int_u^\infty dX$

$$\Rightarrow \int_u^\infty H(X)\frac{1}{(X-u)^{1/2}}\,dX = \int_{X=u}^\infty \int_{x=X}^\infty \frac{dy}{dx}\left(\frac{4x^2}{3}-\frac{20}{3}Xx+\frac{35}{6}X^2\right)\frac{1}{(x-X)^{1/2}(x-u)^{1/2}}\,dx\,dX$$



INTERCHANGING THE ORDER OF INTEGRATION

$$= \int_{x=u}^\infty \int_{X=u}^{x}\left(\frac{4}{3}x^2-\frac{20}{3}Xx+\frac{35}{6}X^2\right)\frac{1}{(x-X)^{1/2}(X-u)^{1/2}}\frac{dy}{dx}\,dX\,dx$$

$$= \int_{x=u}^\infty I(u,x)\frac{dy}{dx}\,dx$$

WHERE THE INNER INTEGRAL

$$I(u,x) = \int_{X=u}^{x}\left(\frac{35}{6}X^2-\frac{20}{3}xX+\frac{4}{3}x^2\right)\frac{1}{(-X^2+(u+x)X-ux)^{1/2}}\,dX$$

$$= \Bigg[\frac{35}{6}\left(\cancelto{0}{\frac{-2X-3(x+u)}{4}(-X^2+(u+x)X-ux)^{1/2}} + \frac{3(u+x)^2-4ux}{8}\int\frac{dX}{(-X^2+(u+x)X-ux)^{1/2}}\right)$$

$$-\frac{20}{3}x\left(\cancelto{0}{\frac{(-X^2+(u+x)X-ux)^{1/2}}{-1}} + \frac{(u+x)}{2}\int\frac{dX}{(-X^2+(u+x)X-ux)^{1/2}}\right)$$

$$+\frac{4}{3}x^2\int\frac{dX}{(-X^2+(u+x)X-ux)^{1/2}}\Bigg]_{X=u}^{x}$$

$$= \left(\frac{3}{16}x^2-\frac{15}{8}ux+\frac{35}{16}u^2\right)\int_u^x\frac{dX}{(x-X)^{1/2}(X-u)^{1/2}}$$

$$= \left(\frac{3}{16}x^2-\frac{15}{8}ux+\frac{35}{16}u^2\right)\cdot\pi$$

$$\Rightarrow \int_u^{\infty} \frac{H(x)}{(x-u)^{1/2}}\,dx = \int_{x=u}^{\infty} \frac{\pi}{16}\left(35u^2 - 30ux + 3x^2\right)\frac{dy}{dx}\,dx$$

DIFFERENTIATING W.R.T. u

$$\Rightarrow \frac{d}{du}\int_u^{\infty} \frac{H(x)}{(x-u)^{1/2}}\,dx = -\frac{\pi}{16}\cdot 8u^2\frac{dy}{du} + \frac{\pi}{16}\int_{x=u}^{\infty}(70u - 30x)\frac{dy}{dx}\,dx$$

AND AGAIN

$$\Rightarrow \frac{d^2}{du^2}\int_u^{\infty} \frac{H(x)}{(x-u)^{1/2}}\,dx = -\pi u\frac{dy}{du} - \frac{\pi}{2}u^2\frac{d^2y}{du^2} - \frac{\pi}{16}\cdot 40u\frac{dy}{du} + \frac{70\pi}{16}\int_{x=u}\frac{dy}{dx}\,dx$$

$$= -\frac{\pi}{2}\left(u^2\frac{d^2y}{du^2} + 7u\frac{dy}{du} + \frac{35}{4}y\right)$$

$$\Rightarrow u^2\frac{d^2y}{du^2} + 7u\frac{dy}{du} + \frac{35}{4}y = -\frac{2}{\pi}\frac{d^2}{du^2}\int_u^{\infty}\frac{H(x)}{(x-u)^{1/2}}\,dx$$

THIS IS AN EULER-TYPE EQUATION, SO LET $u = e^t \Rightarrow u\frac{d}{du} = \frac{d}{dt}$, $u^2\frac{d^2}{du^2} = \frac{d^2}{dt^2} - \frac{d}{dt}$

$$\Rightarrow \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + \frac{35}{4}y = -\frac{2}{\pi}e^{-2t}\left[\frac{d^2}{dt^2} - \frac{d}{dt}\right]\int_{e^t}^{\infty}\frac{H(x)}{(x-e^t)^{1/2}}\,dx$$

$m^2 + 6m + \frac{35}{4} = 0$

$\Rightarrow m_{\pm} = \frac{-6 \pm \sqrt{36-35}}{2}$

$= -\frac{5}{2}, -\frac{7}{2}$

WHICH HAS THE GENERAL SOLUTION

$$y = c_1 e^{-5/2\,t} + c_2 e^{-7/2\,t} + \frac{e^{-\frac{5}{2}t}}{1}\int e^{5/2\,t'}\cdot -\frac{2}{\pi}e^{-2t'}\left[\frac{d^2}{dt'^2} - \frac{d}{dt'}\right]\int_{e^{t'}}^{\infty}\frac{H(x)}{(x-e^{t'})^{1/2}}\,dx\,dt'$$
$$+ \frac{e^{-7/2\,t}}{-1}\int e^{7/2\,t'}\cdot -\frac{2}{\pi}e^{-2t'}\left[\frac{d^2}{dt'^2} - \frac{d}{dt'}\right]\int_{e^{t'}}^{\infty}\frac{H(x)}{(x-e^{t'})^{1/2}}\,dx\,dt'$$

RE-SUBSTITUTING $u = e^t$ AND SUBSTITUTING $u' = e^{t'}$

$$\Rightarrow y = \cancel{c_1 u^{-5/2}} + \cancel{c_2 u^{-7/2}} - \frac{2}{\pi}u^{-5/2}\int^{u} u'^{5/2}\frac{d^2}{du'^2}\left[\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\right]\frac{du'}{u'}$$
$$+ \frac{2}{\pi}u^{-7/2}\int^{u} u'^{7/2}\frac{d^2}{du'^2}\left[\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\right]\frac{du'}{u'}$$

ABSORBED INTO LOWER LIMIT OF u' INTEGRATION

$$\Rightarrow \quad y(u) = \frac{2}{\pi}\left\{ u^{-7/2}\int^{u} \frac{d^2}{du'^2}\left[\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\right]u'^{5/2}\,du' \right.$$

$$\left. - u^{-5/2}\int^{u}\frac{d^2}{du'^2}\left[\int_{u'}^{\infty} \frac{H(x)}{(x-u')^{1/2}}\,dx\right]u'^{3/2}\,du'\right\}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\rho\overline{v_r^4}\right) = \frac{2}{\pi}\left\{ x^{-5/2}\int_{x}^{\infty} u'^{3/2}\frac{d^2}{du'^2}\left[\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\right]du' \right. \qquad x = r^2 \qquad (9)$$

$$\left. - x^{-7/2}\int_{x}^{\infty} u'^{5/2}\frac{d^2}{du'^2}\left[\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\right]du'\right\}$$

INTEGRATING THE FIRST INTEGRAL ON THE RHS BY PARTS

$$\Rightarrow \quad \int_{x}^{\infty} u'^{3/2}\frac{d^2}{du'^2}\left[\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\right]du'$$

$$= \left[u'^{3/2}\cdot\frac{d}{du'}\left[\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\right] - \frac{3}{2}\int u'^{1/2}\frac{d}{du'}\left[\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\right]du'\right]_{u'=x}^{\infty}$$

ASSUMING THINGS ARE WELL-BEHAVED AT INFINITY, AND INTEGRATING BY PARTS AGAIN

$$= -x^{3/2}\frac{d}{dx}\left[\int_{x}^{\infty}\frac{H(X)}{(X-x)^{1/2}}\,dX\right] - \frac{3}{2}\left[u'^{1/2}\cdot\int_{u'}^{\infty}\frac{H(x)\,dx}{(x-u')^{1/2}} - \frac{1}{2}\int u'^{-1/2}\int_{u'}^{\infty}\frac{H(x)}{(x-u')^{1/2}}\,dx\,du'\right]_{u'=x}^{\infty}$$

$$= -x^{3/2}\frac{d}{dx}\left[\int_{x}^{\infty}\frac{H(X)}{(X-x)^{1/2}}\,dX\right] + \frac{3}{2}x^{1/2}\int_{x}^{\infty}\frac{H(X)}{(X-x)^{1/2}}\,dX + \frac{3}{4}\int_{u'=x}^{\infty}u'^{-1/2}\int_{X=u'}^{\infty}\frac{H(X)\,dX\,du'}{(X-u')^{1/2}}$$

BUT, REVERSING THE ORDER OF INTEGRATION ON THE LAST TERM,

$$\int_{u'=x}^{\infty}\int_{X=u'}^{\infty} u'^{-1/2}\frac{H(X)}{(X-u')^{1/2}}\,dX\,du' = \int_{X=x}^{\infty}H(X)\int_{u'=x}^{X}\frac{u'^{-1/2}}{(X-u')^{1/2}}\,du'\,dX$$

$$\cancel{= \int_{X=x}^{\infty}H(X)\left[2\tan^{-1}\sqrt{\frac{X}{X-1}} - 2\tan^{-1}\sqrt{\frac{x}{X-1}}\right]dX}$$

$$= \int_{X=x}^{\infty} H(X) \cdot \left[2 \sin^{-1} \sqrt{\frac{X-x}{X}} \right] dX$$

DOING SIMILAR THINGS WITH THE SECOND TERM,

$$\int_{x}^{\infty} u'^{5/2} \frac{d^2}{du'^2}\left[\int_{u'}^{\infty} \frac{H(X)}{(X-u')^{1/2}}\, dX\right] du' = \left[u'^{5/2} \cdot \frac{d}{du'}\left[\int_{u'}^{\infty} \frac{H(X)}{(X-u')^{1/2}}\, dX\right] - \frac{5}{2}\int u'^{3/2} \frac{d}{du'}\left[\int_{u'}^{\infty} \frac{H(X)}{(X-u')^{1/2}}\, dX\right] du' \right]_{u'=x}^{\infty}$$

$$= -x^{5/2}\frac{d}{dx}\left[\int_{x}^{\infty} \frac{H(X)}{(X-x)^{1/2}}\, dX\right] - \frac{5}{2}\left[u'^{3/2} \int_{u'}^{\infty} \frac{H(X)}{(X-u')^{1/2}}\, dX - \frac{3}{2}\int u'^{1/2} \int_{u'}^{\infty} \frac{H(X)}{(X-u')^{1/2}}\, dX\, du' \right]_{u'=x}^{\infty}$$

$$= -x^{5/2}\frac{d}{dx}\left[\int_{x}^{\infty} \frac{H(X)}{(X-x)^{1/2}}\, dX\right] + \frac{5}{2}x^{3/2}\int_{x}^{\infty} \frac{H(X)}{(X-x)^{1/2}}\, dX + \frac{15}{4}\int_{u'=x}^{\infty} u'^{1/2} \int_{X=u'}^{\infty} \frac{H(X)}{(X-u')^{1/2}}\, dX\, du'$$

AGAIN REVERSING THE ORDER OF INTEGRATION ON THE LAST TERM,

$$\int_{u'=x}^{\infty}\int_{X=u'}^{\infty} u'^{1/2} \frac{H(X)}{(X-u')^{1/2}}\, dX\, du' = \int_{X=x}^{\infty} H(X) \int_{u'=x}^{X} \frac{u'^{1/2}}{(X-u')^{1/2}}\, du'\, dX$$

$$= \int_{X=x}^{\infty} H(X) \left[\sqrt{x(X-x)} + \frac{X}{2} \cdot \left[2 \sin^{-1}\sqrt{\frac{X-x}{X}} \right]\right] dX$$

SUBSTITUTING ALL THIS BACK INTO (9)

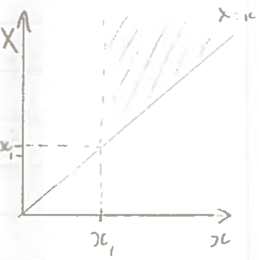
$$\Rightarrow \frac{d}{dx}\rho\overline{v_r^4} = \frac{2}{\pi}\left\{ x^{-5/2}\left[-x^{3/2}\frac{d}{dx}\left(\int_{x}^{\infty} \frac{H(X)}{(X-x)^{1/2}}\, dX\right) + \frac{3}{2}x^{1/2}\int_{x}^{\infty} \frac{H(X)}{(X-x)^{1/2}}\, dX + \frac{3}{2}\int_{x}^{\infty} \sin^{-1}\sqrt{\frac{X-x}{X}}\, H(X)\, dX \right]\right.$$

$$- x^{-7/2}\left[-x^{5/2}\frac{d}{dx}\left(\int_{x}^{\infty} \frac{H(X)}{(X-x)^{1/2}}\, dX\right) + \frac{5}{2}x^{3/2}\int_{x}^{\infty} \frac{H(X)}{(X-x)^{1/2}}\, dX + \frac{15}{4}\int_{x}^{\infty}\left(\sqrt{x(X-x)} + X \sin^{-1}\sqrt{\frac{X-x}{X}}\right) H(X)\, dX\right.$$

$$= \frac{1}{\pi}\int_{X=x}^{\infty}\left\{ -2x^{-2}\frac{H(X)}{(X-x)^{1/2}} - \frac{15}{2}x^{-3}(X-x)^{1/2}H(X) - \left(\frac{15}{2}X - 3x\right)x^{-7/2}\sin^{-1}\sqrt{\frac{X-x}{X}}\, H(X)\right\} dX$$

$$\boxed{\Rightarrow \frac{d}{dx}\rho\overline{v_r^4} = \frac{-1}{2\pi}x^{-7/2}\int_{X=x}^{\infty}\left\{ (15X - 11x)x^{1/2}\frac{H(X)}{(X-x)^{1/2}} + (15X - 6x)\sin^{-1}\sqrt{\frac{X-x}{X}}\, H(X)\right\} dX \qquad (10)}$$

SINCE $\rho\overline{v_r^4} \to 0$ AS $r \to \infty$, WE CAN WRITE



$$\rho\overline{v_r^4}(x_1) = \frac{1}{2\pi}\int_{x=x_1}^{\infty}\int_{X=x}^{\infty} x^{-7/2}\left\{(15X-11x)\,x^{\frac{1}{2}}(X-x)^{-\frac{1}{2}} + (15X-6x)\cos^{-1}\left(\sqrt{\frac{x}{X}}\right)\right\}H(X)\,dX\,dx$$

REVERSING THE ORDER OF INTEGRATION (SIGH!)

$$= \frac{1}{2\pi}\int_{X=x_1}^{\infty} H(X)\int_{x=x_1}^{X}\left\{15X\,x^{-3}(X-x)^{-\frac{1}{2}} - 11\,x^{-2}(X-x)^{-\frac{1}{2}} + 15X\,x^{-7/2}\cos^{-1}\left(\sqrt{\frac{x}{X}}\right) - 6x^{-5/2}\cos^{-1}\left(\sqrt{\frac{x}{X}}\right)\right\}dx\,dX$$

ATTACKING THE UNKNOWN INTEGRALS ONE AT A TIME

$a=-1$
$b=X$

$$\int_{x=x_1}^{X} x^{-3}(X-x)^{-\frac{1}{2}}\,dx = \left[\frac{-3x-2X}{4X^2x^2}(X-x)^{\frac{1}{2}} + \frac{3}{8X^2}\int x^{-1}(X-x)^{-\frac{1}{2}}\,dx\right]_{x_1}^{X}$$

$a=-1$
$b=X$

$$\int_{x=x_1}^{X} x^{-2}(X-x)^{-\frac{1}{2}}\,dx = \left[-\frac{(X-x)^{\frac{1}{2}}}{Xx} + \frac{1}{2X}\int x^{-1}(X-x)^{-\frac{1}{2}}\,dx\right]_{x_1}^{X}$$

$t = x^{\frac{1}{2}}$
$\Rightarrow dx = 2t\,dt,\ x^{-7/2} = t^{-7}$

$$\int_{x=x_1}^{X} x^{-7/2}\cos^{-1}\left(\sqrt{\frac{x}{X}}\right)dx = 2\int_{t=\sqrt{x_1}}^{\sqrt{X}} t^{-6}\cos^{-1}\left(\frac{t}{X^{\frac{1}{2}}}\right)dt$$

$$= 2\left[\frac{t^{-5}}{-5}\cos^{-1}\left(\frac{t}{X^{\frac{1}{2}}}\right) - \frac{1}{5}\int t^{-5}(X-t^2)^{-\frac{1}{2}}\,dt\right]_{\sqrt{x_1}}^{\sqrt{X}}$$

$x = t^2$
$\Rightarrow dt = \frac{1}{2}x^{-\frac{1}{2}}dx$

$$= \frac{2}{5}x_1^{-5/2}\cos^{-1}\left(\sqrt{\frac{x_1}{X}}\right) - \frac{1}{5}\int_{x_1}^{X} x^{-3}(X-x)^{-\frac{1}{2}}\,dx$$

$$\int_{x=x_1}^{X} x^{-5/2}\cos^{-1}\left(\sqrt{\frac{x}{X}}\right)dx = 2\int_{t=\sqrt{x_1}}^{\sqrt{X}} t^{-4}\cos^{-1}\left(\frac{t}{X^{\frac{1}{2}}}\right)dt$$

$$= 2\left[\frac{t^{-3}}{-3}\cos^{-1}\left(\frac{t}{X^{\frac{1}{2}}}\right) - \frac{1}{3}\int t^{-3}(X-t^2)^{-\frac{1}{2}}\,dt\right]_{\sqrt{x_1}}^{\sqrt{X}}$$

$$= \frac{2}{3}x_1^{-3/2}\cos^{-1}\left(\sqrt{\frac{x_1}{X}}\right) - \frac{1}{3}\int_{x_1}^{X} x^{-2}(X-x)^{-\frac{1}{2}}\,dx$$

SUBSTITUTING FOR THE TWO $\cos^{-1}$ INTEGRALS, WE OBTAIN THE INTEGRAL OVER x

$$I_x = 15X\int_{x=x_1}^{X} x^{-3}(X-x)^{-\frac{1}{2}}\,dx - 11\int x^{-2}(X-x)^{-\frac{1}{2}}\,dx + 15X\left[\frac{2}{5}x_1^{-5/2}\cos^{-1}\left(\sqrt{\frac{x_1}{X}}\right) - \frac{1}{5}\int_{x_1}^{X} x^{-3}(X-x)^{-\frac{1}{2}}\,dx\right]$$

$$- 6\left[\frac{2}{3}x_1^{-3/2}\cos^{-1}\left(\sqrt{\frac{x_1}{X}}\right) - \frac{1}{3}\int_{x_1}^{X} x^{-2}(X-x)^{-\frac{1}{2}}\,dx\right]$$

$$= (6X - 4x_1)\,x_1^{-5/2}\cos^{-1}\left(\sqrt{\frac{x_1}{X}}\right) + 12X\int_{x_1}^{X} x^{-3}(X-x)^{-\frac{1}{2}}\,dx - 9\int_{x_1}^{X} x^{-2}(X-x)^{-\frac{1}{2}}\,dx$$

$$= (6X - 4x_1)\,x_1^{-5/2}\cos^{-1}\left(\sqrt{\frac{x_1}{X}}\right) + \frac{3}{Xx_1^2}(3x_1 + 2X)(X-x_1)^{\frac{1}{2}} + \frac{9}{2X}\int_{x_1}^{X} x^{-1}(X-x)^{-\frac{1}{2}}\,dx$$

$$- \frac{9(X-x_1)^{\frac{1}{2}}}{x_1 X} - \frac{9}{2X}\int_{x_1}^{X} x^{-1}(X-x)^{-\frac{1}{2}}\,dx$$

THANKFULLY, THE DIVERGENT INTEGRALS CANCEL

$$\Rightarrow \boxed{\rho\overline{v_r^4}(x) = \frac{1}{\pi}\cdot\frac{1}{x^2}\int_{X=x}^{\infty}\left\{(3X-2x)\,x^{-\frac{1}{2}}\cos^{-1}\left(\sqrt{\frac{x}{X}}\right) + 3(X-x)^{\frac{1}{2}}\right\}H(X)\,dX}$$

FOR A WELL BEHAVED SYSTEM, $\rho\overline{v_r^4}$ IS FINITE AS $x\to 0$, AND THEREFORE THE INTEGRAL MUST VANISH UP TO $O(x^2)$ AS $x\to 0$. SERIES EXPANDING THE INTEGRAND YIELDS

$$\left\{(3X-2x)\,x^{-\frac{1}{2}}\cos^{-1}\left(\sqrt{\frac{x}{X}}\right) + 3(X-x)^{\frac{1}{2}}\right\} \approx \frac{3}{2}\pi X x^{-\frac{1}{2}} - \pi x^{\frac{1}{2}} + O(x^2)$$

SO $\rho\overline{v_r^4}$ WILL REMAIN FINITE AT THE ORIGIN ONLY IF

$$\int_{X=0}^{\infty} X\,H(X)\,dX = 0 \quad \text{AND} \quad \int_{X=0}^{\infty} H(X)\,dX = 0$$

WHICH GIVES THE 2 EXTRA MASS-TO-LIGHT RATIOS (SEE BELOW)

WE NOW NEED A CONVENIENT FORM FOR $H(X)$

WITHOUT MESSING WITH HYDROSTATIC EQUILIBRIUM IN DERIVING (6), WE OBTAIN FOR EQUATION (7) THAT

$$\mu(R)\overline{v^4_{LOS}} = 2\int_R^\infty \left\{ \frac{(r^2-R^2)^{3/2}}{r^3}\rho\overline{v_r^4} + \frac{R^2}{r^2}(r^2-R^2)^{1/2}\left[\frac{d}{dr}(\rho\overline{v_r^4}) + 3\rho\overline{v_r^2}\frac{d\Phi}{dr} + \frac{2\rho}{r}\overline{v_r^4}\right]\right.$$

$$+ \frac{3}{8}\frac{R^4}{r^3}(r^2-R^2)^{-1/2}\left[\frac{1}{3}r^2\frac{d^2}{dr^2}(\rho\overline{v_r^4}) + \frac{7}{3}r\frac{d}{dr}(\rho\overline{v_r^4}) + \frac{8}{3}\rho\overline{v_r^4}\right. \qquad (7')$$

$$\left.\left. + r^2\frac{d\Phi}{dr}\frac{d}{dr}(\rho\overline{v_r^2}) + (5\rho\overline{v_r^2}+\rho\overline{v_t^2})r\frac{d\Phi}{dr} + \rho\overline{v_r^2}r^2\frac{d^2\Phi}{dr^2}\right]\right\}dr$$

$$\Rightarrow \mu(R)\overline{v^4_{LOS}} - 2\int_R^\infty\left\{\frac{3R^2}{r^2}(r^2-R^2)^{1/2}\rho\overline{v_r^2}\frac{d\Phi}{dr} + \frac{3}{8}\frac{R^4}{r^3}(r^2-R^2)^{-1/2}\left[r^2\frac{d\Phi}{dr}\frac{d}{dr}(\rho\overline{v_r^2}) + (5\rho\overline{v_r^2}+\rho\overline{v_t^2})r\frac{d\Phi}{dr}\right.\right.$$

$$\left.\left. + \rho\overline{v_r^2}r^2\frac{d^2\Phi}{dr^2}\right]\right\}dr$$

$$= 2\int_R^\infty\left(\frac{1}{8}\frac{R^4}{r}(r^2-R^2)^{-1/2}\frac{d^2}{dr^2}\rho\overline{v_r^4} + \frac{R^2}{r^2}(r^2-R^2)^{-1/2}\left(r^2-\frac{1}{8}R^2\right)\frac{d}{dr}\rho\overline{v_r^4} + r(r^2-R^2)^{-1/2}\rho\overline{v_r^4}\right)dr \qquad (8')$$

SUBSTITUTING $x=r^2$, $X=R^2$, WE CAN WRITE THE LHS OF (8') AS

$$H(X) = \mu\overline{v^4_{LOS}}(X) - \cancel{2}\int_X^\infty\left\{\frac{3X}{x}(x-X)^{1/2}\rho\overline{v_r^2}F + \frac{3}{8}\frac{X^2}{x^{3/2}}(x-X)^{-1/2}\left[xF.2\cancel{x^{1/2}}\frac{d}{dx}(\rho\overline{v_r^2}) + (5\rho\overline{v_r^2}+\rho\overline{v_t^2})\cancel{x^{1/2}}F\right.\right.$$

$$\left.\left. + \rho\overline{v_r^2}x.2\cancel{x^{1/2}}\frac{dF}{dx}\right]\right\}\frac{dx}{\cancel{2}x^{1/2}}$$

WHERE $F(x) = \frac{d\Phi}{dr} = \frac{GM(x)}{x} = \frac{G\Upsilon L(x)}{x}$

$$\boxed{\Rightarrow H(X) = \mu\overline{v^4_{LOS}}(X) - \int_X^\infty \frac{3}{8}\frac{X^2}{x^{3/2}}(x-X)^{-1/2}\left[\left(8\frac{x}{X}-3\right)\rho\overline{v_r^2}F + \rho\overline{v_t^2}F + 2x\frac{d}{dx}(\rho\overline{v_r^2})F + 2x\rho\overline{v_r^2}\frac{dF}{dx}\right]dx}$$

NOTICE THAT THE FIRST TERM IN $H(X)$ DOES NOT DEPEND ON THE MASS-TO-LIGHT RATIO Υ, WHEREAS THE SECOND TERM IS PROPORTIONAL TO Υ VIA F. (THIS NEGLECTS THE FACT THAT THE MASS-TO-LIGHT RATIO WENT INTO CALCULATING $\rho\overline{v_r^2}$, $\rho\overline{v_t^2}$, BUT THIS CALCULATION IS ONLY BEING MADE TO CHECK FOR CONSISTENCY WITH THAT VALUE).

WE MAY THEREFORE WRITE

$$H(X) = \mu \overline{v^4_{LOS}}(X) - \Upsilon H_\Upsilon(X)$$

FOR A WELL BEHAVED SYSTEM (FOR EXAMPLE, ONE WHERE THE CENTRAL POTENTIAL IS FINITE, AND THE SYSTEM IS OF LIMITED EXTENT), THE 4TH MOMENT OF VELOCITY AT THE SYSTEM CENTER ($x=0$) WILL BE FINITE. WE THEREFORE CONSIDER THE EVALUATION OF (10) FOR SMALL x.

TAYLOR EXPANDING $(X-x)^{\frac{1}{2}}$ AND $\sin^{-1}\left(\sqrt{1-x/X}\right) = \cos^{-1}\left(\sqrt{\frac{x}{X}}\right)$ IN THE INTEGRAND OF (10) ABOUT $x=0$ YIELDS A TERM

$$t = (15X - 11x)x^{\frac{1}{2}}\left(1 + \frac{1}{2}\frac{x}{X} + \frac{3}{8}\left(\frac{x}{X}\right)^2 + O(x^3)\right) + (15X - 6x)\left(\frac{\pi}{2} - \left(\frac{x}{X}\right)^{\frac{1}{2}} - \frac{1}{6}\left(\frac{x}{X}\right)^{\frac{3}{2}} \frac{3}{40}\left(\frac{x}{X}\right)^{\frac{5}{2}} O(x$$

MIRACULOUSLY, THIS SIMPLIFIES TO

$$t = \frac{15\pi}{2}X - 3\pi x + O\left(x^{3/2}\right)$$

AND SO IT IS APPARENT THAT $\frac{d}{dx}(\rho\overline{v_r^2})$ WILL REMAIN FINITE AT $x=0$ ONLY IF

$$\int_{X=0}^{\infty} X H(X)\, dX = 0 \quad \text{AND} \quad \int_{X=0}^{\infty} H(X)\, dX = 0$$

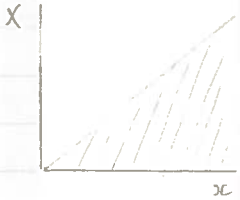
OF THESE, THE SECOND IS MORE USEFUL, AS IT YIELDS THE MASS-TO-LIGHT RATIO

$$\Upsilon = \frac{\int_{X=0}^{\infty} \mu \overline{v^4_{LOS}}\, dX}{\int_{X=0}^{\infty} H_\Upsilon(X)\, dX} = \frac{\frac{1}{\pi} N_{TOT} \left\langle \overline{v^4_{LOS}} \right\rangle}{\int_{X=0}^{\infty} H_\Upsilon(X)\, dX}$$

WHERE N_{TOT} IS THE TOTAL NUMBER OF PARTICLES IN THE SYSTEM AND $\left\langle \overline{v^4_{LOS}} \right\rangle$ IS THE GLOBAL MEAN 4TH MOMENT OF THE LINE-OF-SIGHT VELOCITY DISTRIBUTION, A <u>COMPARATIVELY</u> SIMPLE QUANTITY TO MEASURE. ACTUALLY, MAKES NO DIFFERENCE. USE BOTH!

NOW LET'S PRETTY UP THE TWO M/L INTEGRALS:

1) $$\int_{X=0}^{\infty} H_\gamma(X)\,dX = \int_{X=0}^{\infty}\int_{x=X}^{\infty} \frac{3}{8}\frac{X^2}{x^{3/2}}(x-X)^{-1/2}\left[\left(8\frac{x}{X}-3\right)\rho\overline{v_r^2}F + \rho\overline{v_t^2}F + 2x\frac{d}{dx}\left(\rho\overline{v_r^2}F\right)\right]dx\,dX$$



REVERSING THE ORDER OF INTEGRATION YET AGAIN

$$= \int_{x=0}^{\infty}\int_{X=0}^{x} \frac{3}{8}\frac{X^2}{x^{3/2}}(x-X)^{-1/2}\left[\left(8\frac{x}{X}-3\right)\rho\overline{v_r^2}F + \rho\overline{v_t^2}F + 2x\frac{d}{dx}\left(\rho\overline{v_r^2}F\right)\right]dX\,dx$$

$$= \frac{3}{8}\int_{x=0}^{\infty} \frac{1}{x^{3/2}}\cdot\left[8x\rho\overline{v_r^2}F\cdot\frac{4}{3}x^{3/2} + \left(-3\rho\overline{v_r^2}F+\rho\overline{v_t^2}F + 2x\frac{d}{dx}\left(\rho\overline{v_r^2}F\right)\right)\cdot\frac{16}{15}x^{5/2}\right]dx$$

$$= \int_{x=0}^{\infty}\left[x\left(\frac{14}{5}\rho\overline{v_r^2}F + \frac{2}{5}\rho\overline{v_t^2}F\right) + \frac{4}{5}x^2\frac{d}{dx}\left(\rho\overline{v_r^2}F\right)\right]dx$$

INTEGRATING THE LAST TERM BY PARTS

$$= \int_{x=0}^{\infty}\left[\frac{2}{5}x\left(7\rho\overline{v_r^2}F+\rho\overline{v_t^2}F\right) + \frac{4}{5}\left[-2x\rho\overline{v_r^2}F\right]\right]dx$$

$\beta = 1-\frac{1}{2}\frac{\overline{v_t^2}}{\overline{v_r^2}}$

$$\Rightarrow \int_{X=0}^{\infty} H_\gamma(X)\,dX = \int_{x=0}^{\infty}\frac{2}{5}x F(x)\left(3\rho\overline{v_r^2}+\rho\overline{v_t^2}\right)dx = \int_{x=0}^{\infty}\rho\overline{v_r^2}\left(1-\frac{2}{3}\beta\right)F\,d(r^4)$$

2) $$\int_{X=0}^{\infty} XH_\gamma(X)\,dX = \int_{X=0}^{\infty}X\int_{x=X}^{\infty}\frac{3}{8}\frac{X^2}{x^{3/2}}(x-X)^{-1/2}\left[\left(8\frac{x}{X}-3\right)\rho\overline{v_r^2}F + \rho\overline{v_t^2}F + 2x\frac{d}{dx}\left(\rho\overline{v_r^2}F\right)\right]dx\,dX$$

REVERSING THE ORDER OF INTEGRATION

$$= \int_{x=0}^{\infty}\int_{X=0}^{x}\frac{3}{8}\frac{X^3}{x^{3/2}}(x-X)^{-1/2}\left[\left(8\frac{x}{X}-3\right)\rho\overline{v_r^2}F + \rho\overline{v_t^2}F + 2x\frac{d}{dx}\left(\rho\overline{v_r^2}F\right)\right]dX\,dx$$

$$= \frac{3}{8}\int_{x=0}^{\infty}\frac{1}{x^{3/2}}\left[8x\rho\overline{v_r^2}F\cdot\frac{16}{15}x^{5/2} + \left(-3\rho\overline{v_r^2}F+\rho\overline{v_t^2}F+2x\frac{d}{dx}\left(\rho\overline{v_r^2}F\right)\right)\cdot\frac{32}{35}x^{7/2}\right]dx$$

$$\Rightarrow \int_{x=0}^{\infty} X H_\gamma(x)\, dX = \int_{x=0}^{\infty} \left[x^2 \left(\frac{76}{35} \rho \overline{v_r^2} F + \frac{12}{35} \rho \overline{v_t^2} F \right) + \frac{24}{35} x^3 \frac{d}{dx} \left(\rho \overline{v_r^2} F \right) \right] dx$$

INTEGRATING THE LAST TERM BY PARTS

$$= \int_{x=0}^{\infty} \left[\frac{4}{35} x^2 \left(19 \rho \overline{v_r^2} F + 3 \rho \overline{v_t^2} F \right) - \frac{72}{35} x^2 \rho \overline{v_r^2} F \right] dx$$

$$\Rightarrow \int_{x=0}^{\infty} X H_\gamma(x)\, dX = \int_{x=0}^{\infty} \frac{4}{35} x^2 F(x) \left(\rho \overline{v_r^2} + 3 \rho \overline{v_t^2} \right) dx = \int_{x=0}^{\infty} \frac{4}{15} \rho \overline{v_r^2} \left(1 - 6/7\,\beta\right) F \, d(r^6)$$

TO SUMMARIZE, THE ESTIMATORS FOR $G \times M/L$ ARE:

$$\gamma_0 = \frac{3}{2} \frac{\int_0^\infty R \mu \overline{V_{LOS}^2}\, dR}{\int_0^\infty r \nu L \, dr}$$

$$\gamma_1 = \frac{1}{2} \frac{\int_0^\infty R \mu \overline{V_{LOS}^4}\, dR}{\int_0^\infty r \nu \overline{v_r^2} \left(1 - \frac{2}{5}\beta\right) L \, dr}$$

$$\gamma_2 = \frac{5}{4} \frac{\int_0^\infty R^3 \mu \overline{V_{LOS}^4}\, dR}{\int_0^\infty r^3 \nu \overline{v_r^2} \left(1 - \frac{6}{7}\beta\right) L \, dr}$$